JOURNAL OF ALGEBRA 99, 438–457 (1986)

The Structure of Certain Highest Weight Modules for SL_3

RONALD S. IRVING

Department of Mathematics, University of Washington, Seattle, Washington 98195

Communicated by I. N. Herstein

Received June 27, 1984

1. INTRODUCTION

Let K be an algebraically closed field of characteristic p, with $p > 3$, and let G be the algebraic group $SL_3(K)$. In this paper we determine the structure of the Weyl modules $V(\lambda)$ of G for weights λ in the lowest p^2-alcove, as well as the structure of universal highest weight modules for the restricted enveloping algebra $\mathbf{u}$ of G. The Weyl module results have also been obtained by E. Cline (unpublished) and by Doty and Sullivan [5]. We provide yet another description because ours, being essentially a direct calculation, has the benefit of laying bare the structure in a concrete way. (In turn, our approach would be difficult to extend to other algebraic groups, in contrast to the methods of [5].)

Specifically, for each p-alcove C in the lowest p^2-alcove which is not too close to the walls of the dominant chamber—far enough away so that Jantzen's generic pattern of nine alcoves fits within the dominant chamber—we choose a particular weight λ in C. We then find nine elements in the hyperalgebra U_K of G, each of which sends a generating highest weight vector of $V(\lambda)$ to a vector generating a distinct submodule of $V(\lambda)$ with simple top. Since the composition factors of $V(\lambda)$ are distinct, this furnishes a complete description of the submodules of $V(\lambda)$. By Jantzen's translation principle, the results extend to $V(\lambda')$ for any other weight λ' in C. Section 4 contains these results.

We obtain analogous results in Section 3 for the universal highest weight modules of $\mathbf{u}$, and more generally for the corresponding $\mathbf{u}$-T modules $\hat{Z}(\lambda)$. Following Jantzen, we may view $\hat{Z}(\lambda)$ as a $\mathbf{u}$-T submodule of $V(\lambda)$, and our computations permit a comparison of the two modules. We find, as one may expect, that the structures are the same, but not in a direct way. This is explained in Section 5. We review some background and describe our notation in Section 2.

0021-8693/86 $3.00

I wish to thank J. Ballard, S. Doty, J. Humphreys, and J. Sullivan for conversations on the material of this paper. It was the work of Doty and Sullivan which led me to extend to Weyl modules my earlier results on u-modules.

2. Background

Definitions and additional information for the objects discussed in this section can be found in [4, 6, 9, 11].

2.1.

Let $\mathfrak{g}$ be the complex, semisimple Lie algebra sl_3, with root system R, simple roots $B = \{\alpha, \beta\}$, and nonsimple positive root γ. Thus $\gamma = \alpha + \beta$ and γ is the half-sum of the positive roots. Let $\{\omega_\alpha, \omega_\beta\}$ be a basis of the integral weight lattice X, dual to the coroots $\{\check{\alpha}, \check{\beta}\}$; we will often write (c, d) for the weight $c\omega_\alpha + d\omega_\beta$. Let X^+ be the set of dominant integral weights, let W be the Weyl group, and let W_{p^n} be the affine Weyl group generated by W and translations via $p^n\alpha$ and $p^n\beta$. The group W_{p^n} acts on X via the dot action, with $w \cdot \lambda = w(\lambda + \gamma) - \gamma$ for $w \in W_{p^n}$ and $\lambda \in X$. We define subsets $X_n = \{\lambda \in X^+ \mid (\lambda, \check{\alpha}) < p^n \text{ and } (\lambda, \check{\beta}) < p^n\}$ and $C_n = \{\lambda \in X^+ \mid 0 \leqslant (\lambda, \check{\gamma}) < p^n - 2\}$, so that C_n is simply the lowest alcove in the dominant chamber with respect to the dot action of W_{p^n}, or what we will call the lowest p^n-alcove.

Let $\{x_\alpha, x_\beta, x_\gamma, x_{-\alpha}, x_{-\beta}, x_{-\gamma}, h_\alpha, h_\beta\}$ be the standard Chevalley basis for $\mathfrak{g}$. We will also denote $x_{-\varepsilon}$ by y_ε for a positive root ε. We will denote the Kostant $\mathbb{Z}$-form of the enveloping algebra $U(\mathfrak{g})$ by $U_{\mathbb{Z}}$, the hyperalgebra $K \otimes_{\mathbb{Z}} U_{\mathbb{Z}}$ by U_K, and the restricted enveloping algebra of G by $\mathbf{u}$. Given a root ε, we will denote the element $1 \otimes x_\varepsilon^r/r!$ of U_K by $x_\varepsilon^{(r)}$, and $1 \otimes \binom{h_\varepsilon}{r}$ by $h_{\varepsilon,r}$ for ε in B. We will drop the r if $r = 1$, but it will be clear from the context if the resulting element is in U_K or $U_{\mathbb{Z}}$. All modules are assumed to be finite-dimensional, and G-modules are rational. With this convention, the categories of G-modules and U_K-modules are equivalent [13].

We define the subalgebra H_K of U_K to be generated by $\{h_{\varepsilon,r} \mid \varepsilon \in B, r \in \mathbb{N}^+\}$, and B_K is the subalgebra generated by H_K along with $\{x_\varepsilon^{(r)} \mid \varepsilon \in R^+, r \in \mathbb{N}^+\}$. Let $\hat{\mathbf{u}}$ be the subalgebra $\mathbf{u}H_K$ and $\check{\mathbf{u}}$ the subalgebra $\mathbf{u} \cdot B_K$. Also let $\mathbf{b} = \mathbf{u} \cap B_K$ and $\hat{\mathbf{b}} = \hat{\mathbf{u}} \cap B_K$. A weight λ in X can be viewed as an algebra homomorphism from H_K to K, which we also denote by λ:

$$\lambda(h_{\varepsilon,r}) = \binom{(\lambda, \check{\varepsilon})}{r} \qquad \text{for} \quad \varepsilon \in B.$$

This yields only a few of the possible algebra homomorphisms [see 6], but we will not need the others. The algebras $\hat{\mathbf{u}}$ and $\check{\mathbf{u}}$ take the place of $\mathbf{u}-T$ and $\mathbf{u}-B$ in [11, 12].

2.2.

Given $\lambda \in X^+$, we will denote by $V(\lambda)$ the Weyl module for G with highest weight λ and by $L(\lambda)$ its simple top. Let us also write $V(\lambda)_{\mathbb{C}}$ for the simple $\mathfrak{g}$-module of highest weight λ and $V(\lambda)_{\mathbb{Z}}$ for the lattice $U_{\mathbb{Z}} \cdot v^+$ in $V(\lambda)_{\mathbb{C}}$, where v^+ is a vector of weight λ. Thus $V(\lambda) = U_K \otimes_{U_{\mathbb{Z}}} V(\lambda)_{\mathbb{Z}}$.

For $\lambda \in X$, we denote by $\hat{Z}(\lambda)$ the universal $\hat{\mathbf{u}}$-highest weight module of highest weight λ. Explicitly, let K_λ be the one-dimensional $\hat{\mathfrak{b}}$-module annihilated by x_α and x_β, on which $h_{\varepsilon,r}$ acts as the scalar $\lambda(h_{\varepsilon,r})$. Then

$$\hat{Z}(\lambda) = \hat{\mathbf{u}} \underset{\hat{\mathfrak{b}}}{\otimes} K_\lambda = u \underset{\mathfrak{b}}{\otimes} K_\lambda.$$

It has a simple top $\hat{L}(\lambda)$. Both modules can be viewed as $\check{\mathbf{u}}$-modules by assuming K_λ is the B_K-module annihilated by all elements $\{x_\varepsilon^{(r)} \mid \varepsilon \in B\}$, in which case we will denote them $\check{Z}(\lambda)$ and $\check{L}(\lambda)$.

Given $\nu \in X_1$, the universal highest weight module for $\mathbf{u}$ of weight ν is just $\hat{Z}(\nu)$ as $\mathbf{u}$-module, and we will denote it $Z(\nu)$. Its simple top is the G-simple $L(\nu)$ restricted to $\mathbf{u}$. For arbitrary $\lambda \in X$, the module $\hat{Z}(\lambda)$ on restriction to $\mathbf{u}$ coincides with $Z(\nu)$ for the unique ν in $X_1 \cap (\lambda + pX)$, and $\hat{L}(\lambda)|_{\mathbf{u}} \cong L(\nu)$. However, for $\lambda \in X^+$, the restriction of the G-simple $L(\lambda)$ to $\hat{\mathbf{u}}$ or $\mathbf{u}$ will not be simple unless $\lambda \in X_1$. Explicitly, if $\lambda = p\mu + \nu$ with $\nu \in X_1$, then the Steinberg tensor product theorem implies that $L(\lambda)|_{\hat{\mathbf{u}}}$ is a direct sum of simples $\hat{L}(p\eta + \nu)$, as η runs through the weights of the simple $L(\mu)$. Thus, in the context of $\mathbf{u}$, we will only speak of $L(\nu)$ if ν is in X_1. In turn, for such a ν and $w \in W$, we will write $\overline{w \cdot \nu}$ for the unique weight in $X_1 \cap (w \cdot \nu + pX)$.

2.3.

Jantzen discovered a close relationship between the modules $\hat{Z}(\lambda)$ and $V(\lambda)$ [12, 3.1], a special case of which we state below:

THEOREM. *Given $\lambda \in C_2$, assume all composition factors of $\hat{Z}(\lambda)$ are of the form $L(p\mu + \nu)$ with $\nu \in X_1$ and $\mu \in \bar{C}_1 \cap X^+$. Then*

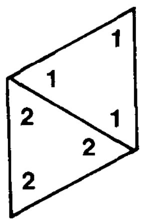
$$(\hat{Z}(\lambda) : \hat{L}(\eta)) = (V(\lambda) : L(\eta))$$

for $\eta \in X_2$.

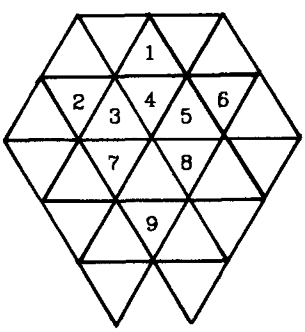
In other words, if $\hat{Z}(\lambda)$ has all its composition factors in the lowest p^2-alcove, then it has the same composition factors as $V(\lambda)$, in the sense that the highest weights occurring correspond.

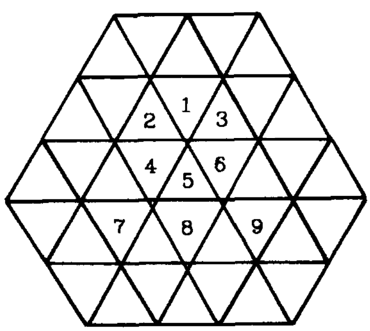
In the setting of the theorem, let v be a highest weight vector of $V(\lambda)$. Then $\mathbf{u} \cdot v$ is necessarily a homomorphic image of $\hat{Z}(\lambda)$ as a $\hat{\mathbf{u}}$-module, and in fact they coincide [12, 6.1], allowing us to study $\hat{Z}(\lambda)$ inside $V(\lambda)$. The same is true for the $\check{\mathbf{u}}$-module $\check{Z}(\lambda)$.

Given ν regular in X_1, the $\mathbf{u}$-module $Z(\nu)$ has nine composition factors, with highest weights in the set $\{\overline{w \cdot \nu} \mid w \in W\}$ [7]. They occur with multiplicity 1 or 2, as indicated in the diagram of X_1 below, depending on whether $\overline{w \cdot \nu}$ is in the upper or the lower of the two p-alcoves [8, p. 23].

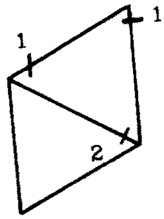


Let $\lambda = p\mu + \nu$ with ν regular in X_1. Then $\hat{Z}(\lambda)$ must have nine composition factors as well, and they occur in nine distinct p-alcoves in one of two patterns, depending on whether or not ν is in C_1. The alcove involved determines the highest weight uniquely, since the weight must lie in $W_p \cdot \lambda$. We depict the two patterns of alcoves below, with 1 placed in the alcove containing λ. The numbering of alcoves indicated will be used throughout the paper. We will distinguish the two types of alcoves by calling them *upward* or *downward*. For instance, 1 is in an upward alcove in the diagram to the left.





In case ν is in X_1 but not regular, either $\nu = (p-1)\gamma$ or ν lies on exactly one wall of the upward alcove. In the former case, $Z(\nu)$ is the Steinberg module $L(\nu)$ and $\hat{Z}(\lambda)$ is simple [7; 8, p. 31]. Otherwise, $Z(\nu)$ has four composition factors, with highest weights $\{\overline{w \cdot \nu} \mid w \in W\}$ and multiplicity 1 or 2 as in the diagram below [8, p. 23].



Let $\lambda = p\mu + \nu$ with the same ν. Then $\hat{Z}(\lambda)$ has four composition factors, with distinct highest weights, and corresponding alcove diagrams which we omit.

In any of the above cases, with $\nu \in X_1$, if in addition μ lies in $\overline{C_1} \cap X^+$ and the resulting alcove pattern lies within the dominant chamber, Jantzen's Theorem implies that $V(\lambda)$ has the same number of composition factors as $\hat{Z}(\lambda)$, with the same highest weights. Let $\mu = c\omega_\alpha + d\omega_\beta$. Then the hypothesis of the theorem is fulfilled in case $c + d < p - 1$ and either

(i) $\nu \in C_1$ and $1 < c, d$ or

(ii) $\nu \notin C_1$ and $1 \leqslant c, d$.

2.4.

Essential to our method is Jantzen's translation principle, of which we only need a special case. Let ν, κ be two weights which either both lie in C_1 or on its upper wall. Then there is an exact functor T_ν^κ on G-modules such that

$$T_\nu^\kappa V(w \cdot \nu) = V(w \cdot \kappa)$$

and

$$T_\nu^\kappa L(w \cdot \nu) = L(w \cdot \kappa)$$

for any $w \in W_p$. These results can be found in [10] or [2]. (The second equality for ν on the upper wall of C_1 is not stated explicitly, but follows easily. For instance, the argument of [2, 2.3] shows that $T_\nu^\kappa L(y \cdot \nu)$ is either (0) or $L(y \cdot \kappa)$ for any $y \in W_p$. But $T_\kappa^\nu T_\nu^\kappa V(w \cdot \nu) = V(w \cdot \nu)$, and the functors are exact, so no simple composition factor of $V(w \cdot \nu)$ can be killed by T_ν^κ).

Analogous results hold for **u**-modules, because of the fact mentioned earlier that the composition factors of $Z(\overline{w \cdot \nu})$ have highest weights in the set $W \cdot \nu + pX$. It follows then by the usual argument, using Jantzen's alcove lemma [2, 1.8], that $T_\nu^\kappa Z(\overline{w \cdot \nu}) = Z(\overline{w \cdot \kappa})$ for $w \in W$. Since $L(\overline{w \cdot \nu})$ is the same simple, for G or **u**, we obtain $T_\nu^\kappa L(\overline{w \cdot \nu}) = L(\overline{w \cdot \kappa})$. Similar remarks apply to $\hat{\mathbf{u}}$-modules.

2.5.

Let M be a finite length module over some algebra, with each composition factor occurring exactly once. Then there is a unique submodule

$M(S)$ of M having S as its top and every submodule of M is a sum of such submodules: $N=\sum_{i\in I} M(S_i)$ if $N/\operatorname{rad} N=\bigoplus_{i\in I} S_i$. This allows one to depict the structure of M by a diagram, with a vertex for each composition factor and an arrow from the vertex for S to the vertex for T in case T is in the top of rad $M(S)$. Submodules of M correspond to subsets of vertices which are closed under images of directed line segments. One can read from the diagram the socle and radical series of M. [See 1 or 3.] We will generally omit the arrowheads in our diagrams, with the understanding that a line segment in a diagram has its head on the lower end.

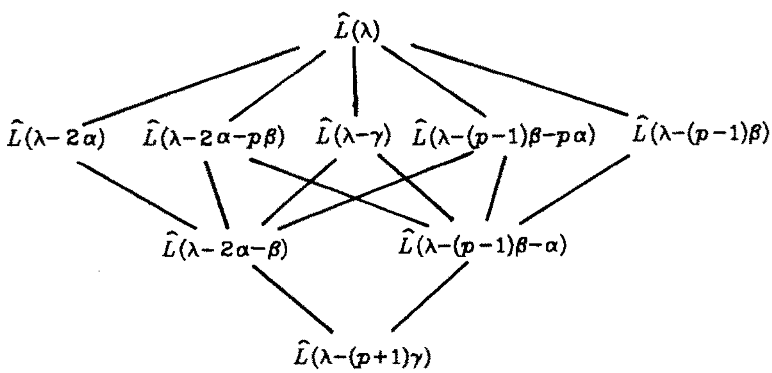
In the setting of 2.3, for λ regular, $V(\lambda)$ has nine distinct composition factors, so its structure can be described by such a diagram. Suppose $\lambda=p\mu+\nu$ as at the end of 2.3 and let κ be in the same alcove as ν. Then the appropriate translation functor $T(=T_\nu^\kappa$ if ν and κ lie in $C_1)$ induces a bijection between the composition factors of $V(\lambda)$ and $V(p\mu+\kappa)$. Since T is exact, it sends $V(\lambda)(S)$ to $V(p\mu+\kappa)(TS)$ for any composition factor S, inducing a lattice isomorphism of the submodule lattices. Therefore it suffices to determine the structure of $V(\lambda)$ for a single λ in each p-alcove. Similar remarks apply in case ν and κ lie on the same wall, and carry over to the modules $\hat{Z}(\lambda)$ for $\hat{\mathbf{u}}$ and $\check{Z}(\lambda)$ for $\check{\mathbf{u}}$.

3. Highest Weight Modules for $\hat{\mathbf{u}}$

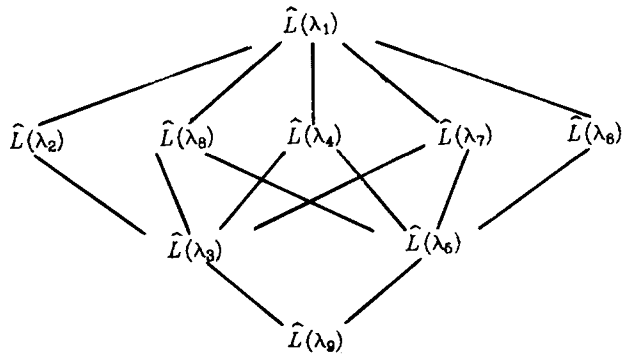
3.1.

We will begin by determining the structure of $\hat{Z}(\lambda)$ for λ a weight in an upward alcove. We can write $\lambda=p(c,d)+\nu$ with ν regular in the upward alcove of X_1, and the translation principle (2.4) permits us to choose any such ν. Take $\nu=(1,p-2)$.

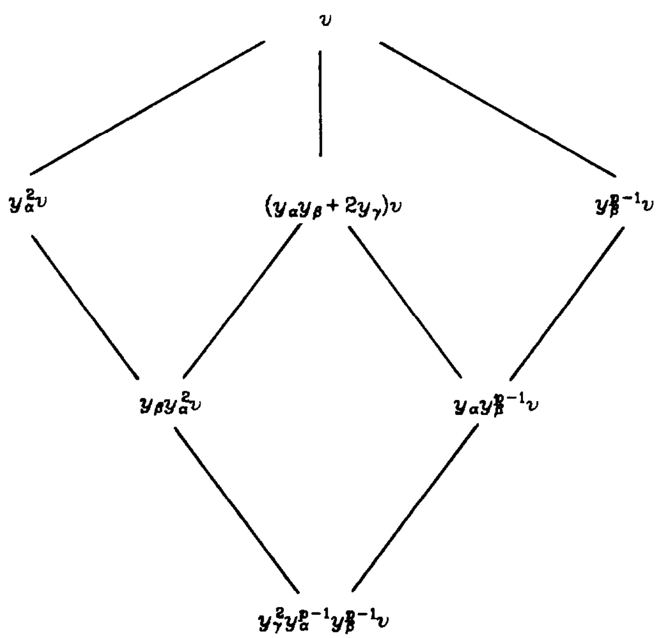
Theorem. *The structure diagram of $\hat{Z}(\lambda)$ for $\lambda=p(c,d)+(1,p-2)$ is given below*:



COROLLARY. *Given a weight λ in an upward alcove, the structure diagram of $\hat{Z}(\lambda)$ is given below, where $\lambda_1 = \lambda$ and λ_i is the unique weight in $W_p \cdot \lambda$ lying in alcove i (with respect to the numbering of* 2.3).



Proof of Theorem. Let v be a vector of weight λ in $\hat{Z}(\lambda)$. Any highest weight vector in $\hat{Z}(\lambda)$ of weight η must generate a submodule which is a homomorphic image of $\hat{Z}(\eta)$, and so has a simple top. In the diagram below, we list highest weight vectors in $\hat{Z}(\lambda)$, connecting two by a descending sequence of line segments if the lower vector lies in the submodule generated by the higher vector. Since the set $\{y_\alpha^i y_\beta^j y_\gamma^k v \mid 0 \leqslant i, j, k < p\}$ is a basis for $\hat{Z}(\lambda)$, weight considerations show that no additional lines can be drawn. To see that all the lines indicated should be drawn, note that $y_\beta y_\alpha^2 v = y_\alpha(y_\alpha y_\beta + 2y_\gamma) v$ and $y_\alpha y_\beta^{p-1} v = y_\beta^{p-2}(y_\alpha y_\beta + 2y_\gamma) v$.



Let $s = y_\gamma y_\beta^{p-1} y_\alpha v$ and $t = y_\gamma y_\alpha^{p-1} y_\beta^{p-2} v$. Along with the seven vectors above, these yield nine vectors of precisely the weights listed in the claimed structure diagram. If we show that $\hat{\mathbf{u}}s$ is a module with simple top, such that s is a highest weight vector in $\hat{\mathbf{u}}s/\mathrm{rad}\ \hat{\mathbf{u}}s$, and the analogous result for $\hat{\mathbf{u}}t$, we will have accounted for nine distinct composition factors; by 2.3 there are no more. The relative location in the structure diagram of the nine simples will depend on the relative relationship of the nine vectors, and it will suffice to show:

(i) The submodule N generate by $y_\alpha^2 v$, $y_\beta^{p-1} v$, and $(y_\alpha y_\beta + 2y_\gamma) v$ does not contain s or t.

(ii) The submodule $\hat{\mathbf{u}}s$ contains $y_\beta y_\alpha^2 v$ and $y_\alpha y_\beta^{p-1} v$ but not $y_\alpha^2 v$, $y_\beta^{p-1} v$, or $(y_\alpha y_\beta + 2y_\gamma) v$.

(iii) The image of s is a highest weight vector in $\hat{\mathbf{u}}s/(\hat{\mathbf{u}}y_\beta y_\alpha^2 v + \hat{\mathbf{u}}y_\alpha y_\beta^{p-1} v)$, but no other homomorphic image of $\hat{\mathbf{u}}s$ contains a highest weight vector of weight $\lambda - 2\alpha - p\beta$.

(ii′), (iii′) The same statements as (ii) and (iii) with s replaced bt t, and $\lambda - 2\alpha - p\beta$ replaced by $\lambda - (p-1)\beta - p\alpha$.

Statements (i), (ii), and (ii′) describe the relative relationships of the nine vectors, while (iii) and (iii′) provide the required information for $\hat{\mathbf{u}}s$ and $\hat{\mathbf{u}}t$. Explicitly, by (ii) and the first part of (iii), $\hat{\mathbf{u}}s$ has a copy of $\hat{L}(\lambda - 2\alpha - p\beta)$ in its top and has length 4. If another simple is in the top, then we must find a highest weight vector of weight $\lambda - 2\alpha - p\beta$ in $\hat{\mathbf{u}}s/\hat{\mathbf{u}}y_\beta y_\alpha^2 v$ or $\hat{\mathbf{u}}s/\hat{\mathbf{u}}y_\alpha y_\beta^{p-1} v$, which the second part of (iii) rules out. Similarly for $\hat{\mathbf{u}}t$.

To verify (i), consider the $(\lambda - 2\alpha - p\beta)$-weight space of N. It must be spanned by $y_\gamma y_\alpha (y_\beta^{p-1} v)$, $y_\alpha y_\beta^{p-1} (y_\alpha y_\beta + 2y_\gamma) v$, and $y_\gamma y_\beta^{p-2} (y_\alpha y_\beta + 2y_\gamma) v$. But these vectors are all equal, and distinct from s. For t, we observe that the $(\lambda - (p-1)\beta - p\alpha)$-weight space of N has as basis

$$\{ y_\gamma^{p-1-r} y_\alpha^{r+1} y_\beta^r v + (r+1) y_\gamma^{p-r} y_\alpha^r y_\beta^{r-1} v \mid 1 \leqslant r \leqslant p-1 \},$$

and so cannot contain t.

For (ii) and the first part of (iii), we can compute the following:

$$x_\alpha s = y_\beta (y_\alpha y_\beta^{p-1} v), \qquad x_\beta x_\alpha s = y_\alpha y_\beta^{p-1} v,$$

$$x_\beta s = y_\beta^{p-2} (y_\beta y_\alpha^2 v), \qquad x_\beta^{p-1} s = y_\beta y_\alpha^2 v.$$

It is obvious that $\hat{\mathbf{u}}s$ cannot contain the three vectors generating N. Similarly, (ii′) and the first part of (iii′) follow from

$$x_\alpha t = -y_\alpha^{p-2} (y_\alpha y_\beta^{p-1} v), \qquad x_\alpha^{p-1} t = -y_\alpha y_\beta^{p-1} v,$$

$$x_\beta t = -2y_\alpha y_\beta y_\alpha^{p-1} y_\beta^{p-3} v \in \hat{\mathbf{u}}y_\beta y_\alpha^2 v, \qquad x_\alpha^{p-2} x_\beta^{p-3} t = -\tfrac{1}{2} y_\beta y_\alpha^2 v.$$

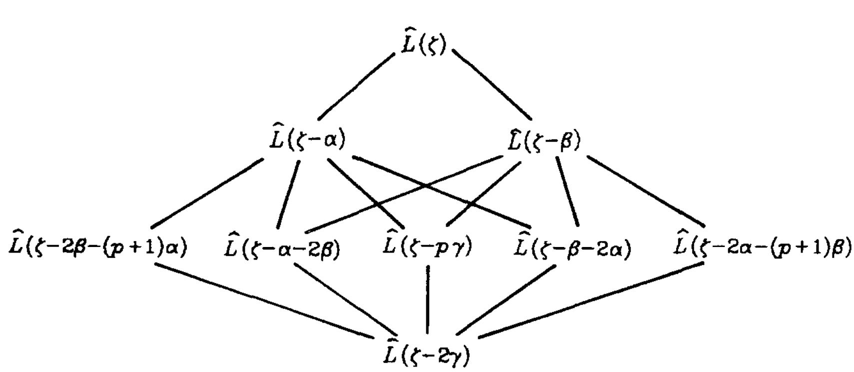
For the rest of (iii), we first note that the image of s is the only vector of weight $\lambda - 2\alpha - p\beta$ in $\hat{\mathbf{u}}s/\hat{\mathbf{u}}y_\alpha y_\beta^{p-1}v$, but is not a highest weight vector. In $\hat{\mathbf{u}}s/\hat{\mathbf{u}}y_\beta y_\alpha^2 v$, we also have the image of $y_\gamma y_\alpha y_\beta^{p-1}v$, which is killed by x_α and sent to $y_\alpha^2 y_\beta^{p-1}v$ by x_β. Hence no linear combination of s and $y_\gamma y_\alpha y_\beta^{p-1}v$ is killed by x_α modulo $y_\beta y_\alpha^2 v$.

It remains to verify the second part of (iii′). The only vector of the desired weight in $\hat{\mathbf{u}}t/\hat{\mathbf{u}}y_\beta y_\alpha^2 v$ is the image of t, which is not a highest weight vector. We need only check that no linear combination of t and the vectors in $\{ y_\gamma^{p-2-i} y_\alpha^i y_\beta^i (y_\beta y_\alpha^2 v) \mid 0 \leqslant i \leqslant p-2\}$ is a highest weight vector modulo $y_\alpha y_\beta^{p-1}v$. Note that the vectors in brackets are not linearly independent, since the vector $n = \sum_{i=0}^{p-2} c_i y_\gamma^{p-2-i} y_\alpha^i y_\beta^i (y_\beta y_\alpha^2 v)$ is 0 if the coefficients satisfy $ic_i + c_{i-1} = 0$ for $0 < i \leqslant p-2$, as one sees by straightening the monomials. On the other hand, assuming $x_\alpha(n + at) = 0$ for some a in K, and suitable coefficients c_i, and defining d_i in K by the formula $x_\alpha n = \sum_{i=0}^{p-1} d_i y_\gamma^{p-1-i} y_\alpha^i y_\beta^i v$, we find that d_i must be 0 for $0 \leqslant i \leqslant p-2$. Computing $x_\alpha n$ and straightening the monomials, we find that this implies $ic_i + c_{i-1} = 0$ for $0 < i \leqslant p-2$, so $n = 0$. This completes the proof. ∎

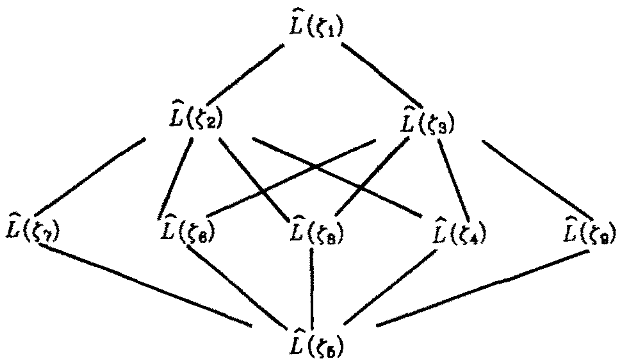
3.2.

Note that by reversing the roles of α and β in 3.1, we obtain symmetric information for $Z(\lambda')$ with $\lambda' = p(c, d) + (p-2, 1)$. With this in mind, the case of a downward alcove is easy to treat.

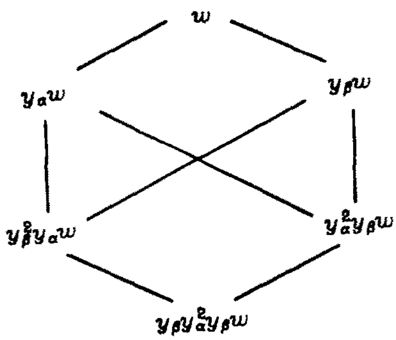
THEOREM. *Let $\zeta = p(c, d) + (0, 0)$. Then $\hat{Z}(\zeta)$ has the structure diagram below*:



COROLLARY. *Given a weight ζ in a downward alcove, the structure diagram of $\hat{Z}(\zeta)$ is given below, where $\zeta_1 = \zeta$ and ζ_i is the unique weight in $W_p \cdot \zeta$ lying in alcove i (with respect to the numbering of 2.3).*



Proof of Theorem. We begin again with a diagram of highest weight vectors, taking w to be a vector of weight ζ in $\hat{Z}(\zeta)$:



Since $y_\alpha w$ and $y_\beta w$ are highest weight vectors, we obtain hornomorphisms of $\hat{Z}(\zeta-\alpha)$ and $\hat{Z}(\zeta-\beta)$ into $\hat{Z}(\zeta)$. But $\zeta-\alpha=p(c-1, d)+(p-2, 1)$ and $\zeta-\beta=p(c, d-1)+(1, p-2)$, so the results of 3.1 apply to the submodules generated by $y_\alpha w$ and $y_\beta w$. Using $y_\beta w$, we obtain vectors $s=y_\gamma y_\beta^{p-1} y_\alpha y_\beta w$ and $t=y_\gamma y_\alpha^{p-1} y_\beta^{p-1} w$ which generate submodules with simple top. By symmetry, $y_\gamma y_\alpha^{p-1} y_\beta y_\alpha \omega$ and $y_\gamma y_\beta^{p-1} y_\alpha^{p-1} w$ also generate submodules with simple top. This yields ten such vectors, of nine distinct weights. Since $\hat{Z}(\zeta)$ has only nine composition factors, $y_\gamma y_\alpha^{p-1} y_\beta^{p-1} w$ and $y_\gamma y_\beta^{p-1} y_\alpha^{p-1} w$ must generate the same submodule, as one could check directly (showing they are multiples modulo $y_\beta y_\alpha^2 y_\beta w$). Sorting out the vectors leads to the diagram below for their relative location and thence to the desired structure diagram.

$$
\begin{tikzcd}[column sep=small]
 & & w \arrow[dl, dash] \arrow[dr, dash] & & \\
 & y_\alpha w \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] \arrow[drr, dash] & & y_\beta w \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
y_\gamma^2 y_\alpha^{p-1} w \arrow[drr, dash] & y_\beta^2 y_\alpha w \arrow[dr, dash] & y_\gamma y_\alpha^{p-1} y_\beta^{p-1} w \arrow[d, dash] & y_\alpha^2 y_\beta w \arrow[dl, dash] & y_\gamma^2 y_\beta^{p-1} w \arrow[dll, dash] \\
 & & y_\beta y_\alpha^2 y_\beta w & &
\end{tikzcd}
$$

Remark. The Theorem also follows from 3.1 by duality. The dual vector space M^* of a $\hat{\mathfrak{u}}$-module M can be given a $\hat{\mathfrak{u}}$-module structure via the transpose anti-automorphism, with $(x_\varepsilon f)(m) = f(x_{-\varepsilon} m)$ for $f \in M^*$, $m \in M$, and $\varepsilon \in R$. Under this definition, the dual of $\hat{Z}(\lambda)$ is a highest weight module with respect to the opposite choice of simple roots $B^0 = \{-\alpha, -\beta\}$. Given $\lambda = p(c, d) + (1, p-2)$, the highest weight of $\hat{Z}(\lambda)^*$, in terms of B^0, is $p(p-c+1,\ p-d+1) + (p-3, 0)$, and one can check by computing the highest weights of duals of the various simples, with respect to B^0, that the structure diagram of Corollary 3.2 results.

3.3.

Let us also explicitly determine the structure of $\hat{Z}(\eta)$ for η nonregular. Hence η has the form $p\mu + \nu$ with ν on the boundary of the upward alcove in X_1. We may assume $\nu \neq (p-1, p-1)$ (see 2.3), so by the translation principle (2.4) it suffices to consider a single ν from each of the three walls. By symmetry, we need only handle one upper wall of the alcove, so we may assume $\nu = (p-2, 0)$ or $\nu = (0, p-1)$.

THEOREM. *Let* $\eta = p(c, d) + (p-2, 0)$. *Then* $\hat{Z}(\eta)$ *has the structure diagram below*:

$$
\begin{tikzcd}
 & \hat{L}(\eta) \arrow[dl, dash] \arrow[dr, dash] & \\
\hat{L}(\eta - \beta) \arrow[dr, dash] & & \hat{L}(\eta - (p-1)\alpha) \arrow[dl, dash] \\
 & \hat{L}(\eta - p\gamma) &
\end{tikzcd}
$$

Proof. Let z be a vector of weight η in $\hat{Z}(\eta)$. Then the vectors $y_\beta z$, $y_\alpha^{p-1} z$, and $y_\gamma y_\beta^{p-1} y_\alpha^{p-1} z$ are highest weight vectors of distinct weights. The result follows once we note that $y_\gamma y_\beta^{p-1} y_\alpha^{p-1} z = (\sum_{i=1}^{p-1} (p-1-i)!\, y_\gamma^{p-i} y_\alpha^i y_\beta^{i-1}) y_\beta z$. ∎

3.4.

THEOREM. *Let* $\eta = p(c, d) + (0, p-1)$. *Then* $\hat{Z}(\eta)$ *has the structure diagram below*:

$$\begin{tikzcd}
 & \hat{L}(\eta) \arrow[dl, dash] \arrow[dr, dash] & \\
\hat{L}(\eta-\alpha) \arrow[dr, dash] & & \hat{L}(\eta-\alpha-p\beta) \arrow[dl, dash] \\
 & \hat{L}(\eta-\gamma) &
\end{tikzcd}$$

Proof. Let u be a vector of weight η in $\hat{Z}(\eta)$. Then $y_\alpha u$ and $y_\beta y_\alpha u$ are highest weight vectors. Consider $m = y_\beta^{p-1} y_\gamma u$. We have

$$x_\alpha m = 0,$$

$$x_\beta m = y_\beta^{p-1} y_\alpha u, \qquad x_\beta^{p-1} m = y_\beta y_\alpha u.$$

Thus m is a highest weight vector modulo $y_\beta y_\alpha u$, and since m is not in the submodule $\hat{\mathbf{u}} y_\alpha u$, the indicated diagram results. ∎

3.5.

We have made the assumption that $p > 3$ because otherwise there are no weights in the lowest p^2-alcove satisfying the conditions of 2.3—all the weights are too close to the walls of the dominant chamber. However, this restriction is inessential for the structure of $\hat{\mathbf{u}}$ highest weight modules. For $p = 2$, the weight v must be one of three non-regular weights: $(0, 0)$, $(0, 1)$, or $(1, 0)$, and 3.3 and 3.4 apply. For $p = 3$, there are two regular weights in X_1, the weights $(0, 0)$ and $(1, 1)$, and these are exactly the choices of v used in 3.1 and 3.2 ($(1, p-2) = (1, 1)$). As is well known [8, p. 26], on restriction to $\mathbf{u}$, the modules $Z(\lambda)$ and $Z(\eta)$ behave differently than in higher characteristic, having only two distinct composition factors, with multiplicity 6 and 3. But the structures are really the same as in characteristic $p > 3$, as is evident at the $\hat{\mathbf{u}}$-level where the degeneracy in the weights available in X_1 disappears.

4. Weyl Modules for G

4.1.

We begin by recording an elementary fact.

LEMMA. *Let λ be in X^+ and let q, r be positive integers with $q \leqslant (\lambda, \check{\alpha})$ and $r \leqslant (\lambda, \check{\beta})$. Let v be a vector of weight λ in $V(\lambda)$. Then the $(\lambda - q\alpha - r\beta)$-weight space of $V(\lambda)$ has as basis $\{ y_\gamma^{(i)} y_\alpha^{(q-i)} y_\beta^{(r-i)} \mid 0 \leqslant i \leqslant \min\{q, r\}\}$.*

Proof. By the definition of $V(\lambda)$, it suffices to prove this for $V(\lambda)_{\mathbb{C}}$. Let $M(\lambda)$ be the Verma module for $\mathfrak{g}$ of highest weight λ, with a vector v^+ of weight λ. Then $V(\lambda)_{\mathbb{C}} = M(\lambda)/\operatorname{rad} M(\lambda)$ and $\operatorname{rad} M(\lambda)$ is generated by $y_\alpha^{q+1} v^+$ and $y_\beta^{r+1} v^+$, from which the result follows. ∎

We wish to analyze Weyl modules with highest weight in the lowest p^2-alcove, and we can use the information of Section 3 as a head start. Let us begin with nonregular weights, in order to see what happens in simpler cases first.

THEOREM. *Let $\eta = p(c, d) + (p-2, 0)$ and assume $c + d < p - 1$, $c > 0$, $d > 0$. The structure diagram of $V(\eta)$ is given below:*

$$\begin{tikzcd}[column sep=small]
 & L(\eta) \arrow[dl, dash] \arrow[dr, dash] & \\
L(\eta - \beta) \arrow[dr, dash] & & L(\eta - (p-1)\alpha) \arrow[dl, dash] \\
 & L(\eta - p\gamma) &
\end{tikzcd}$$

Proof. Let z be a vector of weight η in $V(\eta)$. Following 3.3, we consider the vectors $y_\beta z$, $y_\alpha^{p-1} z$, and $y_\gamma y_\beta^{p-1} y_\alpha^{p-1} z$, which are nonzero in $V(\eta)$ by the Lemma. We saw in 3.3 that they are highest weight vectors for $\hat{\mathfrak{u}}$, but they are also U_K-highest weight vectors. The only question is whether $x_\alpha^{(p)}$ and $x_\beta^{(p)}$ annihilate $y_\gamma y_\beta^{p-1} y_\alpha^{p-1} z$, which we can verify by computing in $V(\eta)_{\mathbb{Z}}$. For instance, letting z^+ be a vector of weight η in $V(\eta)_{\mathbb{Z}}$, we have

$$x_\alpha^p y_\gamma y_\beta^{p-1} y_\alpha^{p-1} z = -(p!)(pc)(pc+1) \cdots (pc+p-2) y_\beta^p z^+,$$

so

$$\frac{x_\alpha^p}{p!} (y_\gamma y_\beta^{p-1} y_\alpha^{p-1} z) = -(pc) \cdots (pc+p-2) y_\beta^p z^+$$

and passing to $V(\lambda)$ yields 0. By 2.3, $V(\eta)$ has four composition factors, and we have found them all. ∎

4.2.

THEOREM. *Let $\eta = p(c,d) + (0, p-1)$ and assume $c+d < p-1$, $c>0$, $d>0$. The structure diagram of $V(\eta)$ is given below:*

$$\begin{tikzcd} & L(\eta) \arrow[dl, dash] \arrow[dr, dash] & \\ L(\eta-\alpha) \arrow[dr, dash] & & L(\eta-\alpha-p\beta) \arrow[dl, dash] \\ & L(\eta-\gamma) & \end{tikzcd}$$

Proof. Let u be a vector of weight η in $V(\eta)$. It is obvious that the $\hat{\mathfrak{u}}$-highest weight vectors $y_\alpha u$ and $y_\beta y_\alpha u$ are U_K-highest weight vectors. By 2.3 and 3.4, there is one composition factor remaining to be located, of weight $\eta - \alpha - p\beta$. In $\hat{Z}(\eta)$, the missing composition factor is the top of the length two submodule $\hat{\mathfrak{u}} \cdot m$, where $m = y_\gamma y_\beta^{p-1} u$. But computing in $V(\eta)_{\mathbb{Z}}$, with highest weight vector u^+, we find that

$$x_\beta^p y_\gamma y_\beta^{p-1} u^+ = p!(dp+1)\cdots(dp+p-1)\, y_\alpha u^+,$$

so $x_\beta^{(p)} m = (p-1)!\, y_\alpha u = -y_\alpha u$ in $V(\eta)$. Thus $U_K \cdot m$ contains $y_\alpha u$, in contrast to $\hat{\mathfrak{u}} m$.

Instead, consider the vector $m' = (d+1)\, y_\gamma y_\beta^{p-1} u + y_\beta^{(p)} y_\alpha u$. It is readily verified that this is killed by x_α, $x_\alpha^{(p)}$, $x_\beta^{(p)}$ and sent by x_β to $d y_\beta^{p-1} y_\alpha u$. (Of course, $x_\alpha^{(r)}$ and $x_\beta^{(r)}$ kill m' for larger r by weight considerations.) For instance, to verify that $x_\beta^{(p)} m' = 0$, we compute in $V(\eta)_{\mathbb{Z}}$ again:

$$x_\beta^p y_\beta^p y_\alpha u^+ = p!\,(dp+1)\cdots(dp+p)\, y_\alpha u^+,$$

and

$$x_\beta^{(p)} y_\beta^{(p)} y_\alpha u = \frac{(dp+1)\cdots(dp+p-1)(d+1)}{(p-1)!}\, y_\alpha u = (d+1)\, y_\alpha u.$$

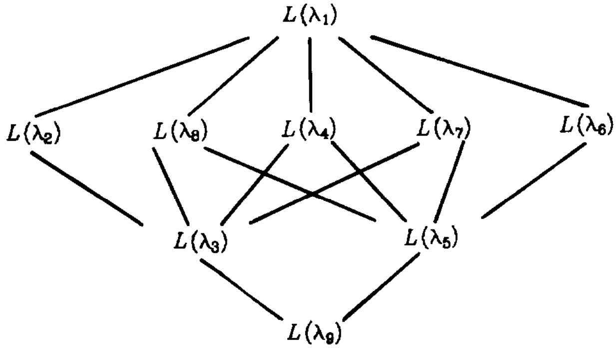
Since $x_\beta^{p-2} m' = y_\beta y_\alpha u$, we see that $U_K \cdot m'$ is a submodule of length two, with top $L(\eta - \alpha - p\beta)$ and socle $L(\eta - \gamma)$, accounting for all the submodules with simple top. ▮

Remark. By the translation principle, the results of 4.1 and 4.2 carry over to other nonregular weights $p(c,d) + v'$, with $c+d<p-1$ and $c>0$, $d>0$.

4.3.

The phenomenon which arose in the proof of 4.2 recurs for regular weights.

THEOREM. *Let $\lambda = p(c, d) + \nu$ and assume that $c + d < p - 1$, that $c > 0$ and $d > 0$, and that ν is in the upward alcove of X_1. The structure diagram of $V(\lambda)$ is given below, where $\lambda_1 = \lambda$ and λ_i is the unique weight of $W_p \cdot \lambda$ in the* ith *alcove* (*with respect to the numbering of* 2.3).



Proof. By the translation principle we may set $\nu = (1, p-2)$, and by 2.3 there are nine composition factors in $V(\lambda)$, with the same highest weights as those of $\hat{Z}(\lambda)$. Let v be a vector of weight λ in $V(\lambda)$. The seven vectors listed in the proof of 3.1 as highest weight vectors of $\hat{Z}(\lambda)$ may be viewed as non-zero vectors in $V(\lambda)$ by Lemma 4.1. They are also U_K-highest weight vectors in $V(\lambda)$, the only question being whether $x_\alpha^{(p)}$ and $x_\beta^{(p)}$ annihilate $y_\gamma^2 y_\alpha^{p-1} y_\beta^{p-1} v$. This can be checked as usual by computing in $V(\lambda)_{\mathbb{Z}}$.

Let s and t be the vectors $y_\gamma y_\beta^{p-1} y_\alpha v$ and $y_\gamma y_\alpha^{p-1} y_\beta^{p-2} v$, as in 3.1. Obviously $x_\alpha^{(p)} s = x_\beta^{(p)} t = 0$, and computing in $V(\lambda)_{\mathbb{Z}}$ yields

$$x_\beta^{(p)} s = -y_\alpha^2 v, \; x_\alpha^{(p)} t = y_\beta^{p-1} v.$$

Thus, the submodules $U_K \cdot s$ and $U_K \cdot t$ contain highest weight vectors which $\hat{\mathfrak{u}} s$ and $\hat{\mathfrak{u}} t$ do not contain. As in the proof of 4.2, we must modify s and t. Let $s' = (d+1)s + y_\beta^{(p)} y_\alpha^2 v$ and let $t' = (c+1)t - y_\alpha^{(p)} y_\beta^{p-1} v$. Then calculation yields

$$x_\alpha s' = (d+1) y_\beta (y_\alpha y_\beta^{p-1} v), \qquad x_\beta s' = d y_\beta^{p-2} (y_\beta y_\alpha^2 v),$$

$$x_\alpha t' = -c y_\alpha^{p-2} (y_\alpha y_\beta^{p-1} v), \qquad x_\beta t' = -2(c+1) y_\alpha y_\beta y_\alpha^{p-1} y_\beta^{p-3} v,$$

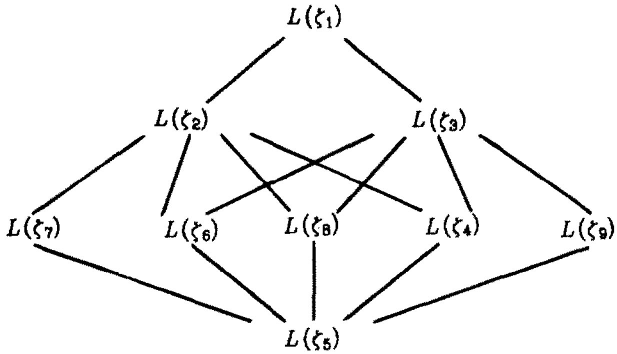
and $x_\alpha^{(p)}$ and $x_\beta^{(p)}$ annihilate s' and t'.

Referring to the proof of 3.1, it suffices to verify the analogues of (i), (ii), (ii′), (iii), and (iii′) with s and t replaced by s' and t'. The above equalities yield (ii) and (ii′), plus the first part of (iii) and (iii′). For (i), we use the argument of 3.1. The $(\lambda - \alpha - p\beta)$-weight space of the submodule of $V(\lambda)$ generated by $y_\alpha^2 v$, $y_\beta^{p-1} v$, and $(y_\alpha y_\beta + 2y_\gamma) v$ is spanned by the vectors in 3.1, all of which equal $y_\gamma y_\alpha y_\beta^{p-1} v$, along with $y_\beta^{(p)} y_\alpha^2 v$. If s' lies in this space, then so does $s' - y_\beta^{(p)} y_\alpha^2 v$, or s. Using Lemma 4.1, we obtain a contradiction. The analogous argument works for t', since the $(\lambda - (p-1)\beta - p\alpha)$-weight space of $V(\lambda)$ has as basis the vectors listed in 3.1 and $y_\alpha^{(p)} y_\beta^{p-1} v$. If t' lies in this space, t lies in the space indicated in 3.1, a contradiction. Similarly, the proof of the rest of (iii) and (iii$'$) carries over from 3.1 to this setting, with a single additional vector to take into account in each case. ▮

4.4.

The case of a downward alcove is easily handled, in the same way that we passed from 3.1 to 3.2.

THEOREM. *Let $\zeta = p(c, d) + \nu$ and assume that $c + d < p - 1$, that $c > 1$ and $d > 1$, and that ν is in C_1. The structure diagram of $V(\zeta)$ is given below, where $\zeta_1 = \zeta$ and ζ_i is the unique weight of $W_p \cdot \zeta$ is the* ith *alcove (with respect to the numbering of* 2.3).



Proof. By the translation principle, we may set $\nu = (0, 0)$, and by 2.3 there are nine composition factors in $V(\zeta)$, with the same highest weights as those of $\hat{Z}(\zeta)$. Let w be a vector of weight ζ in $V(\zeta)$. By Lemma 4.1 and weight considerations, the six vectors listed in the proof of 3.2 are non-zero U_K-highest weight vectors in $V(\zeta)$. In particular, the submodules $U_K y_\alpha w$ and $U_K y_\beta w$ are homomorphic images of $V(\zeta - \alpha)$ and $V(\zeta - \beta)$. Using the proof of 4.3 for $V(\zeta - \beta)$, and the symmetric results for $V(\zeta - \alpha)$, we find that the following vectors generate length two submodules with simple top and $U_K y_\beta y_\alpha^2 y_\beta w$ as socle:

$$(dy_\gamma y_\beta^{p-1} y_\alpha + y_\beta^{(p)} y_\alpha^2) y_\beta w,$$

$$(cy_\gamma y_\alpha^{p-1} y_\beta + y_\alpha^{(p)} y_\beta^2) y_\alpha w,$$

and

$$y_\gamma y_\alpha^{p-1} y_\beta^{p-1} w.$$

For instance, the first vector is sent by x_β to $(d-1)\,y_\beta^{p-2}\,(y_\beta\, y_\alpha^2\, y_\beta w)$ and killed by x_α, $x_\alpha^{(p)}$, and $x_\beta^{(p)}$. From the nine vectors, we obtain the claimed structure diagram. ∎

5. Comparison of Modules for $\hat{\mathbf{u}}$ and G

5.1.

Let λ be a weight in the lowest p^2-alcove to which Theorem 2.3 applies and let v be a vector of weight λ in $V(\lambda)$. As noted in 2.3, the $\hat{\mathbf{u}}$-submodule $\hat{\mathbf{u}}\cdot v$ of $V(\lambda)$ is isomorphic to $\hat{Z}(\lambda)$, and Theorem 2.3 states that the composition factors have the same highest weights. The results of Sections 3 and 4 show in addition that the structure diagrams of $V(\lambda)$ and $\hat{Z}(\lambda)$ are isomorphic, in a manner compatible with the correspondence of highest weights of composition factors. The most natural explanation for this situation would be that the lattice homomorphism θ from the lattice $\mathbf{T}$ of submodules of $V(\lambda)$ to the lattice $\mathbf{R}$ of submodules of $\hat{Z}(\lambda)$ defined by $\theta(N)=N\cap\hat{\mathbf{u}}v$ is an isomorphism, with inverse ψ given by $\psi(N')=U_K N'$, but our results show otherwise.

Perhaps the clearest way to view this situation is via $\check{\mathbf{u}}$-modules. Recall that $\check{\mathbf{u}}$ is the subalgebra $\mathbf{u}B_K$ between $\hat{\mathbf{u}}$ and U_K, and that $\hat{Z}(\lambda)$ is also the universal $\check{\mathbf{u}}$-highest weight module, in which role it is denoted $\check{Z}(\lambda)$. Let S be the lattice of submodules of $\check{Z}(\lambda)$. Then θ and ψ factor through S, yielding maps as in the diagram below, defined in the obvious way.

$$\mathbf{R}\xrightarrow{\psi_1}\mathbf{S}\xrightarrow{\psi_2}\mathbf{T}$$

$$\mathbf{T}\xrightarrow{\theta_1}\mathbf{S}\xrightarrow{\theta_2}\mathbf{R}.$$

For any choice of λ, the maps ψ_1 and θ_1 are surjective, and ψ_2 and θ_2 are injective, but $\mathbf{S}$ generally has smaller size than $\mathbf{R}$ or $\mathbf{T}$. We can best picture the images of ψ and θ via $\mathbf{S}$, which we now describe. From its description, the claimed properties of ψ_i and θ_i for $i=1$ or 2 will follow.

5.2.

Theorem. *Let $\eta=p(c,d)+(0,p-1)$. The $\check{\mathbf{u}}$-module $\check{Z}(\eta)$ is uniserial, with structure diagram below:*

$$\begin{array}{c}\check{L}(\eta)\\ |\\ \check{L}(\eta-\alpha-p\beta)\\ |\\ \check{L}(\eta-\alpha)\\ |\\ \check{L}(\eta-\gamma).\end{array}$$

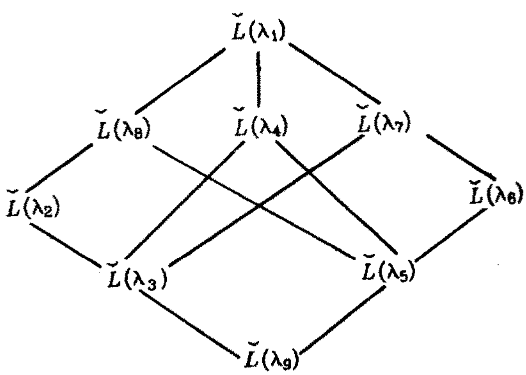
Proof. We adopt the notation of 3.4 and 4.2. As in 4.2, the vectors u, $y_\alpha u$, and $y_\beta y_\alpha u$ are highest weight vectors, and $x_\beta^{(p)} m = -y_\alpha u$. (In 4.2 we computed this last equality in $V(\eta)_{\mathbb{Z}}$. Here we must use the relations in U_K instead.) Recall from 3.4 that $x_\alpha m = 0$ and $x_\beta m = y_\beta^{p-1} y_\alpha u$. Thus the image of m in $\check{\mathbf{u}} \cdot m / \check{\mathbf{u}} \cdot y_\alpha u$ is a highest weight vector, and $\check{\mathbf{u}} \cdot m$ must have length three. Since m is the only vector in $\check{Z}(\eta)$ of weight $\eta - \alpha - p\beta$, up to scalar multiple, $\check{\mathbf{u}} \cdot m$ must be the unique submodule with simple top $\check{L}(\eta - \alpha - p\beta)$, yielding the claimed structure diagram. ∎

Remark. Loosely speaking, we can describe the situation in the following way. In $\hat{Z}(\eta)$, the vectors m and $y_\alpha u$ are independent in the sense that each generates a submodule not containing the other. In passing to $V(\eta)$, for η as in 4.2, the presence of $x_\beta^{(p)}$ destroys this independence, but the availability of $y_\beta^{(p)}$ allows us to replace m by $m' = (d+1)\, m + y_\beta^{(p)} y_\alpha u$, thereby restoring independence and the same structure. However, in $\check{Z}(\eta)$ we are not free to replace m by m', and must accept the dependence of $y_\alpha u$ on m. Regarding the lattice maps of 5.1, we see that ψ_1 sends both $\hat{\mathbf{u}} m$ and $\hat{\mathbf{u}} m + \hat{\mathbf{u}} y_\alpha u$ to $\check{\mathbf{u}} m$, and ψ_2 send this to $U_K m' + U_K y_\alpha u$. The submodule $U_K m'$ is not in the image of ψ.

5.3.

The same phenomenon occurs for regular weights. We will only deal with the case of an upward alcove, from which one easily obtains the analogous result for a downward alcove.

THEOREM. *Given a weight λ in an upward alcove, the structure diagram of $\check{Z}(\lambda)$ is given below, where $\lambda_1 = \lambda$ and λ_i is the unique weight in $W_p \cdot \lambda$ lying in alcove i (with respect to the numbering of 2.3).*



Proof. By the translation principle, we may assume $\nu = (1, p-2)$. Adopting the notation of 3.1, we obtain seven highest weight vectors for $\check{\mathbf{u}}$

in $\check{Z}(\lambda)$, as in 4.3. The data of 3.1 and 4.3 carry over to $\check{Z}(\lambda)$, revealing that the submodule $\check{u}s$ contains $y_\alpha^2 v$ and $y_\alpha y_\beta^{p-1} v$, while $\check{u}t$ contains $y_\beta^{p-1} v$ and $y_\beta y_\alpha^2 v$; in addition s and t become highest weight vectors in the homomorphic images of $\check{u}s$ and $\check{u}t$ modulo the respective pairs of vectors. This information yields the indicated structure diagram, provided we check that no other homomorphic image of $\check{u}s$ has a highest weight vector of weight $\lambda - 2\alpha - p\beta$, and analogously for t. But this follows immediately from the proofs of the second parts of (iii) and (iii′) in 3.1. ∎

5.4.

Remark. Assume λ satisfies the hypotheses of 4.3. We can see that the submodules of $\hat{Z}(\lambda)$ with top $\hat{L}(\lambda_8)$ and top $\hat{L}(\lambda_8) \oplus \hat{L}(\lambda_2)$ have the same image under ψ_1 and ψ, and similarly for the submodules with top $\hat{L}(\lambda_7)$ and $\hat{L}(\lambda_7) \oplus \hat{L}(\lambda_6)$. Let us describe the situation in another way. Given a composition series $(0) = N_0 \subset \cdots \subset N_9 = \hat{Z}(\lambda)$, the map ψ sends it to a composition series for $V(\lambda)$ if and only if, among the composition factors, $\hat{L}(\lambda_2)$ occurs below $\hat{L}(\lambda_8)$ and $\hat{L}(\lambda_6)$ occurs below $\hat{L}(\lambda_7)$. Otherwise, for each of these pairs of simples in the opposite order, there will be indices $i < j$ with $\psi N_i = \psi N_{i+1}$ and $\psi N_{j+1} / \psi N_j$ equal to the direct sum of the two simples. This follows directly from an examination of the vectors $y_\alpha^2 v$, s, s' and $y_\beta^{p-1} v$, t, t' in the proofs of 3.1, 4.3, and 5.3. For a general semisimple algebraic group G, a discussion of the passage of composition series from a module $\check{Z}(\lambda)$ to $V(\lambda)$ can be found in [12, Sect. 3].

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